647510 **MSC4MatC402x** **Seat No :________**

M.Sc.(Maths.) Semester - 4 (*CBCS*) Examination
March/April - 2024 (NEW COURSE)
Integration Theory (Core)

Time: 02:30 Hours **Marks: 70**

Instructions:

1. All questions are compulsory.
2. Figures to the right indicate marks.

Q.1.(A) Answer the following question. **(04)**

(1) Let μ be a measure on a measurable space $(X, \mathcal{A})$. If E and F are measurable subsets of X, $E \subset F$ and if $\mu(F) < \infty$, then show that $\mu(E - F) = \mu(E) - \mu(F)$.

(2) Let f be an integrable function on a measure space $(X, \mathcal{A}, \mu)$. Then show that $\left|\int_X f\, d\mu\right| \leq \int_X |f|\, d\mu$.

Q.1.(B) Answer any two questions out of three. **(10)**

(1) Let $(X, \mathcal{A}, \mu)$ be a measurable space, and let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions on X converging to f(pointwise) on X. Then show that $\int_E f\, d\mu = \lim\limits_{n\to\infty} \int_E f_n\, d\mu$.

(2) Let $\{f_n\}$ be a sequence of nonnegative measurable functions on a measure space $(X, \mathcal{A}, \mu)$ converging to f(pointwise) on X. If $f_n \leq f$ for all n, then $\int_X f\, d\mu = \lim\limits_{n} \int_X f_n\, d\mu$.

(3) State and prove Lebesgue Dominated Convergence theorem.

Q.2.(A) Answer the following question. **(04)**

(1) Given an example to show that a measurable set with positive measure is not always a positive set.

(2) Let v be a signed measure on a measurable space $(X, \mathcal{A})$, and let A and B be negative sets. Then show that $A \cup B$ is a negative set.

Q.2.(B) Answer any two questions out of three. **(10)**

(1) Let $\{A, B\}$ and $\{A_1, B_1\}$ be Hahn decomposition of a signed measure space $(X, \mathcal{A}, v)$. Then show that $A \Delta A_1$ and $B \Delta B_1$ are null sets.

(2) Let v be a signed measure on a measurable space $(X, \mathcal{A})$. Show that there is a pair of measures v^+ and v^- such that $v = v^+ + v^-$ and $v^+ \perp v^-$.

(3) Let v be a signed measure on a measurable space $(X, \mathcal{A})$, then show that the Jordan decomposition of v is unique.

Q.3.(A) Answer the following question. **(04)**

(1) Let v_1 and v_2 be finite signed measure on a measurable space $(X, \mathcal{A})$, and μ be a measure on $(X, \mathcal{A})$. If $v_1 \ll \mu$ and $v_2 \ll \mu$, then show that $\alpha v_1 + \alpha v_2 \ll \mu$.

(2) Define Radon Nikodym derivative.

Q.3.(B) Answer any two questions out of three. **(10)**

(1) State and prove Radon Nikodym theorem.

(2) State and prove Lebesgue Decomposition theorem.

(3) Let v_1 and v_2 be measures and μ be a $\sigma-$ finite measure on a measurable space $(X, \mathcal{A})$, and let $v_1 \ll \mu$ and $v_2 \ll \mu$. Then show that $\left[\frac{d(v_1+v_2)}{d\mu}\right] = \left[\frac{dv_1}{d\mu}\right] + \left[\frac{dv_2}{d\mu}\right]$.

Q.4.(A) **Answer the following question.** **(04)**

(1) If f is bounded a.e. $[\mu]$ on measurable space $(X, \mathcal{A})$, then $|f(x)| \leq \|f\|_\infty$ a.e. $[\mu]$ on X.

(2) Let $(X, \mathcal{A}, \mu)$ be a measure space, let f be a measurable bounded a.e. $[\mu]$ on X. Then show that $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

Q.4.(B) **Answer any two questions out of three.** **(10)**

(1) Let $(X, \mathcal{A}, \mu)$ be a measure space, let f and g be a measurable function on X, and let $1 \leq p < \infty$. Then show that

$$\left(\int_X |f + g|^p \, d\mu\right)^{\frac{1}{p}} \leq \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}} + \left(\int_X |g|^p \, d\mu\right)^{\frac{1}{p}}$$

(2) Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and let $1 \leq p < r < \infty$. Then show that $L^r(\mu) \subset L^p(\mu)$.

(3) Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $1 \leq p < \infty$. Then show that $(L^p(\mu), \|\cdot\|_p)$ is a Banach space.

Q.5.(A) **Answer the following question.** **(04)**

(1) State Riesz-Markov theorem.

(2) Let X be a locally compact Hausdorff space, and let M be a $\sigma -$ algebra on X containing all Baire sets. When is $E \in M$ called outer regular for μ ?

Q.5.(B) **Answer any two questions out of three.** **(10)**

(1) Let μ be a finite measure defined on a $\sigma -$ algebra M which contains all the Baire sets of a locally compact space X. If μ is inner regular, then show that it is regular.

(2) Let μ^* be a topologically regular outer measure on a locally compact Hausdorff space X. Show that each Borel sets is $\mu^* -$ measurable.

(3) Let μ be a Baire measure on a locally compact space X, and let E be a $\sigma -$ bounded Baire set in X. If $\epsilon > 0$, then show that there is a $\sigma -$ compact open set O containing E such that $\mu(O - E) < \epsilon$.

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